Parametric Fuzzy Implications Produced Via Classes of Strong Negations

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Purpose

The purpose of this paper is the creation of new parametric fuzzy implications via the two main fuzzy connectives, N-negations and T-norms. The N-negations used are the N^{λ}, N^{ω} and N^{α} and the conjunctions are the T_M, T_P and T_{LK}.

Motivation

The produced parametric fuzzy implications as well as the strategy used to create them offer more flexibility and speed in comparison to other methods of generating fuzzy implications and their products.

Introduction

Fuzzy implications generalizations were created. Bustince, Burillo and Soria in 2003 [1], as well as Callejas, Marcos and Bedregal in 2012, created a strategy which constructs any fuzzy implication, using automorphism functions [2]. In 2020 and 2021 the survey on fuzzy implications generalization was continued with the papers of Makariadis, Souliotis and Papadopoulos [3], [4]. In 2022 the paper of Makariadis-Papadopoulos [5] focused on fuzzy implications generalization through automorphism functions.

Introduction

The present paper aims at constructing generator fuzzy implications with the composition of strong fuzzy negations and strict fuzzy t-norms, bearing in mind the relevant papers ([3] and [4]). This gives the possibility of changing the implication with one parameter, giving thus an algorithmic procedure of the implication. The tools for the construction of generator fuzzy implications are the classes of strong negations, as shown in table [1], the strict fuzzy t-norms, as shown in table [2] and the equation linking them (Baczynski book, page 87 [6]). The generator fuzzy implications which were produced are shown in table [3].

Preliminaries (Fuzzy Negations)

Some definitions retrieved from the literature can be found in the following references: (Baczyński M., 1.4.1–1.4.2 Definitions, pp. 13–14, [6]), (Bedregal B.C., p. 1126, [7]), (Fodor J., 1.1–1.2 Definitions, p. 3, [8]), (Gottwald S., 5.2.1 Definition, p. 85, [9]), (Weber S., 3.1 Definition, p. 121, [10]) and (Trillas E., p. 49, [11]).

Definition

A function $N : [0,1] \to [0,1]$ is called a Fuzzy negation if

$(N1): \quad N(0) = 1, \ \ N(1) = 0;$

$(N2): \quad N \textit{ is decreasing}.$

A fuzzy negation N is called strict if, in addition to the former properties, the following apply:

(N3) : N is strictly decreasing;

(N4) : N is continuous.

A fuzzy negation N is called strong if the following property is satisfied:

$(N5): \ \ N(N(x)) = x, \ \ x \in [0,1].$

The following table presents three well-known families of fuzzy negations. Those fuzzy negations can be found in the work by Baczyński M., p. 15, [6].

Table: Basic fuzzy negations classes.

Designation	**Equation**
Sugeno class	$N^{\lambda}(x) = \frac{1-x}{1+\lambda x}, \ \ \lambda \in (-1, +\infty)$
Yager class	$N^{\omega}(x) = (1 - x^{\omega})^{\frac{1}{\omega}}, \ \ \omega \in (0, +\infty)$
Souliotis-Papadopoulos class	$N^{\alpha}(x) = \sqrt{(a^2 - 1)x^2 + 1} + \alpha \cdot x, \ \ a \leq 0$

Triangular Norms (Conjunctions)

The following definition can be found in: (Klement E.P et al., 1.1 Definition, pp. 4–10, [12]), (Baczyński M., 2.1.1, 2.1.2 Definitions, pp. 41–42, [6]) and (Weber S., 2.1 Definition, pp. 116–117, [10]).

Definition

A function $T : [0,1]^2 \to [0,1]$ is called a triangular norm, shortly, t-norm, if it satisfies, for all $x, y \in [0,1]$, the following conditions:

$(T1): \quad T(x,y) = T(y,x), \quad (commutativity);$

$(T2): \quad T(x, T(y,z)) = T(T(x,y), z), \quad (associativity);$

$(T3): \quad if \;\; y \le z, \;\; then \;\; T(x,y) \le T(x,z), \quad (monotonicity);$

$(T4): \quad T(x,1) = x, \quad (boundary \;\; condition).$

In the following table, three well-known t-norms are presented. Those t-norms can be found in: (Baczyński M., p. 42, [6]).

Table: Basic t-norms.

Designation	**Equation**
Minimum	$T_M(x, y) = \min\{x, y\}$
Algebraic product	$T_p(x, y) = x \cdot y$
Lukasiewicz	$T_{LK}(x, y) = \max(x + y - 1, 0)$

Fuzzy Implications

The fuzzy implication functions are probably some of the main functions in fuzzy logic. They play a similar role to that played by classical implications in crisp logic. The fuzzy implication functions are used to execute any fuzzy "if-then" rule on fuzzy systems.
The following definition can be found: (Baczyński M., p. 2, [6]) and (Fodor J., p. 299, [13]).

Definition

A binary operator $I : [0,1]^2 \rightarrow [0,1]$ is said to be an implication function, or an implication, if, for all $x, y \in [0,1]$, it satisfies:

$(I1) : I(x,\ z) \geq I(y,\ z)$ *when* $x \leq y$, *the first place antitonicity*;

$(I2) : I(x,\ y) \leq I(x,\ z)$ *when* $y \leq z$, *the second place isotonicity*;

$(I3) : I(0,0) = 1$, *boundary condition*;

$(I4) : I(1,1) = 1$, *boundary condition*;

$(I5) : I(1,0) = 0$, *boundary condition*.

Math and Equations

The equation: I (x,y) =N(T (x,N (y))), (see Corollary 2.5.31, p. 87, [6]) is a composition of the two most well known connectives, the N-negations and the T-norms. If in the N-negations's place strong negation classes (N^{λ}, N^{ω} and N^{α}) are used and in the T-norms's place fuzzy conjunctions (T_M, T_P and T_{LK}) are used, then parametric fuzzy implication generators are produced. Formulas [1, 2 and 3] can be used to generate the new I-implications.

Theorem 1

Assume the following:

1. A $N : [0,1] \to [0,1]$ strong negation function, which can be replaced with the known from the literature fuzzy negations
   - $N^{\lambda}(x) = \frac{1-x}{1+\lambda x}, \quad \lambda > -1$
   - $N^{\omega}(x) = \sqrt[\omega]{1 - x^{\omega}}, \ \ \omega > 0$
   - $N^{\alpha}(x) = \sqrt{(a^2 - 1)x^2 + 1} + \alpha \cdot x, \ \ a \leq 0$

Theorem 1

2. A continuous Archemedean and strict t-norm $T : [0,1]^2 \rightarrow [0,1]$, which can be replaced with the known from the literature fuzzy conjunctions
   - $T_M(x, y) = \min\{x, y\}$
   - $T_P(x, y) = x \cdot y$
   - $T_{LK}(x, y) = \max\{x + y - 1, 0\}$

Theorem 1

Then, there is a function $I_\lambda : [0,1]^2 \to [0,1]$ which is a I-implication, such that:

$$I_\lambda(x,y) = N^\lambda(T(x, N^\lambda(y))) \quad (1)$$

Then, there is a function $I_\omega : [0,1]^2 \to [0,1]$ which is a I-implication, such that:

$$I_\omega(x,y) = N^\omega(T(x, N^\omega(y))) \quad (2)$$

Then, there is a function $I_\alpha : [0,1]^2 \to [0,1]$ which is a I-implication, such that:

$$I_\alpha(x,y) = N^\alpha(T(x, N^\alpha(y))) \quad (3)$$

Proposition 1

Assume the following:

A $I_\lambda : [0,1]^2 \to [0,1]$ fuzzy implication function, which is defined by equation [1]

1. A conjugation function $T_M : [0,1]^2 \to [0,1]$, which is defined: $T_M(x,y) = \min\{x,y\}$
   Then, there is a function $I_\lambda^1 : [0,1]^2 \to [0,1]$ which is a I-implication, such that:

$$I_\lambda^1(x,y) = \frac{1 - \min\left\{x, \frac{1-y}{1+\lambda \cdot y}\right\}}{1 + \lambda \cdot \min\left\{x, \frac{1-y}{1+\lambda \cdot y}\right\}} \tag{4}$$

Proposition 1

2. A conjugation function $T_P : [0,1]^2 \to [0,1]$, which is defined: $T_P(x,y) = x \cdot y$
   Then, there is a function $I_\lambda^2 : [0,1]^2 \to [0,1]$ which is a I-implication, such that:
$$I_\lambda^2(x,y) = \frac{1 - x \cdot \frac{1-y}{1+\lambda \cdot y}}{1 + \lambda \cdot x \cdot \frac{1-y}{1+\lambda \cdot y}} \tag{5}$$

Proposition 1

3. A conjugation function $T_{LK} : [0,1]^2 \to [0,1]$, which is defined: $T_{LK}(x,y) = \max\{x+y-1, 0\}$
   Then, there is a function $I_\lambda^3 : [0,1]^2 \to [0,1]$ which is a I-implication, such that:

$$I_\lambda^3(x,y) = \frac{1 - \max\left\{x + \frac{1-y}{1+\lambda\cdot y} - 1, 0\right\}}{1 + \lambda\cdot\max\left\{x + \frac{1-y}{1+\lambda\cdot y} - 1, 0\right\}} \tag{6}$$

The graph [Figure 1] shows the fuzzy implications I_λ^1, I_λ^2 and I_λ^3 constructed via the equations [4, 5 and 6] using N^λ and T_M, T_P and T_{LK}.

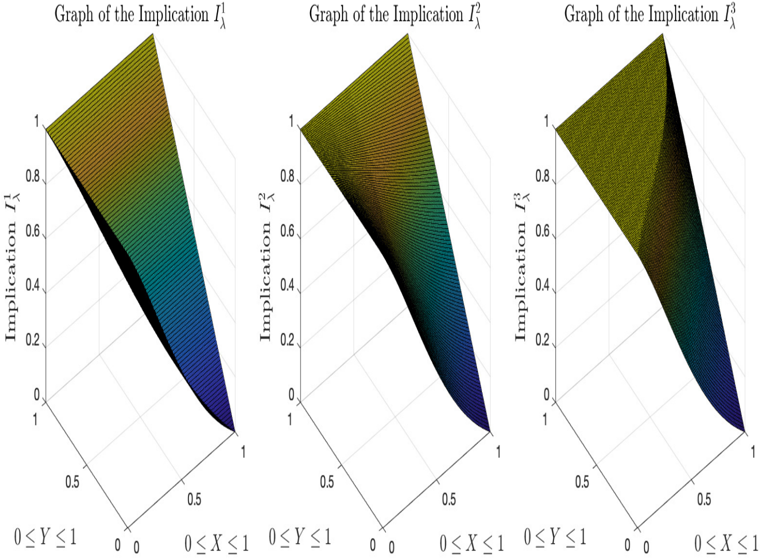

Figure: Graph of the Implications I_λ^1, I_λ^2 and I_λ^3

Proposition 2

Assume the following:
A $I_\omega : [0,1]^2 \to [0,1]$ fuzzy implication function, which is define by equation [2]

1. A conjugation function $T_M : [0,1]^2 \to [0,1]$, which is defined: $T_M(x,y) = \min\{x,y\}$
   Then, there is a function $I_\omega^4 : [0,1]^2 \to [0,1]$ which is a I-implication, such that:

$$I_\omega^4(x,y) = \sqrt[\omega]{1 - \left(\min\left\{x, \sqrt[\omega]{1-y^\omega}\right\}\right)^\omega} \tag{7}$$

Proposition 2

2. A conjugation function $T_P : [0,1]^2 \to [0,1]$, which is defined:
   $T_P(x,y) = x \cdot y$
   Then, there is a function $I^5_\omega : [0,1]^2 \to [0,1]$ which is a I-implication, such that:

$$I^5_\omega(x,y) = \sqrt[\omega]{1 - x^\omega + (x \cdot y)^\omega} \tag{8}$$

Proposition 2

3. A conjugation function $T_{LK} : [0,1]^2 \to [0,1]$, which is defined: $T_{LK}(x,y) = \max\{x+y-1, 0\}$
Then, there is a function $I_\omega^6 : [0,1]^2 \to [0,1]$ which is a I-implication, such that:

$$I_\omega^6(x,y) = \sqrt[\omega]{1 - \left(\max\left\{x + \sqrt[\omega]{1-y^\omega} - 1, 0\right\}\right)^\omega} \qquad (9)$$

The graph [Figure 2] shows the fuzzy implications I_{ω}^{4}, I_{ω}^{5} and I_{ω}^{6} constructed via the equations [7, 8 and 9] using N^{ω} and T_M, T_P and T_{LK}.

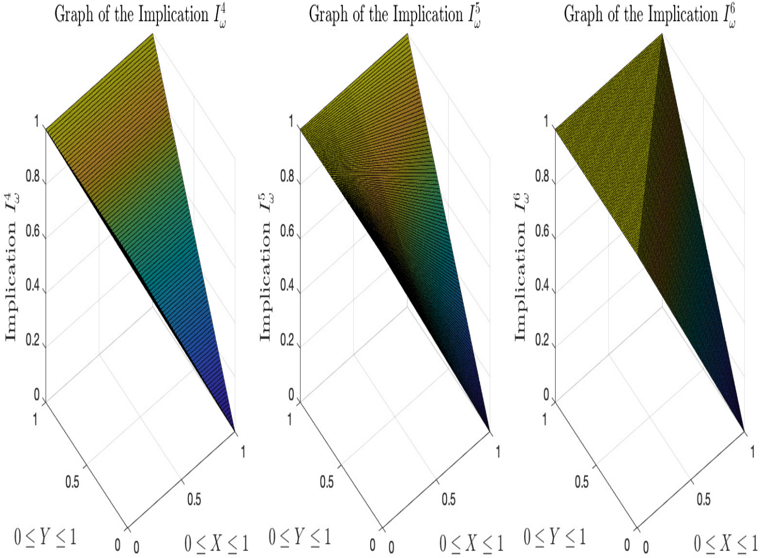

Figure: Graph of the Implications I_ω^4, I_ω^5 and I_ω^6

Proposition 3

Assume the following:
A $I_\alpha : [0,1]^2 \to [0,1]$ fuzzy implication function, which is defined by equation [3]

1. A conjugation function $T_M : [0,1]^2 \to [0,1]$, which is defined: $T_M(x,y) = \min\{x,y\}$
   Then, there is a function $I_\alpha^7 : [0,1]^2 \to [0,1]$ which is a I-implication, such that:

$$I_a^7(x,y) = \sqrt{(a^2-1).\left(\min\left\{x, \sqrt{(a^2-1)y^2+1}+\alpha\cdot y\right\}\right)^2+1} \\ +a\cdot\min\left\{x, \sqrt{(a^2-1)y^2+1}+\alpha\cdot y\right\} \tag{10}$$

Proposition 3

2. A conjugation function $T_P : [0,1]^2 \to [0,1]$, which is defined: $T_P(x,y) = x \cdot y$
   Then, there is a function $I_\alpha^8 : [0,1]^2 \to [0,1]$ which is a I-implication, such that:

$$I_a^8(x,y) = \sqrt{(a^2-1)\cdot x^2 \cdot \left(\sqrt{(a^2-1)\,y^2+1}+\alpha\cdot y\right)^2+1} \\ +a\cdot x\cdot\left(\sqrt{(a^2-1)\,y^2+1}+\alpha\cdot y\right) \quad (11)$$

Proposition 3

3. A conjugation function $T_{LK} : [0,1]^2 \to [0,1]$, which is defined: $T_{LK}(x,y) = \max\{x + y - 1, 0\}$
   Then, there is a function $I_{\alpha}^{9} : [0,1]^2 \to [0,1]$ which is a I-implication, such that:

$$I_a^9(x,y) = \sqrt{(a^2-1).\left(\max\left\{x + \sqrt{(a^2-1)y^2+1} + \alpha \cdot y - 1, 0\right\}\right)^2 -}$$
$$a \cdot \max\left\{x + \sqrt{(a^2-1)y^2+1} + \alpha \cdot y - 1, 0\right\} \quad (12)$$

The graph [Figure 3] shows the fuzzy implications I_{α}^{7}, I_{α}^{8} and I_{α}^{9} constructed via the equations [10, 11 and 12] using N^{α} and T_M, T_P and T_{LK}.

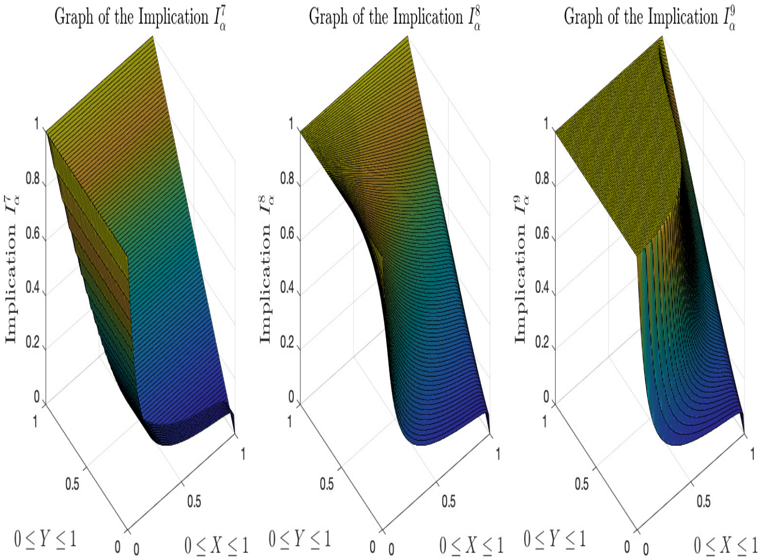

Figure: Graph of the Implications I_α^7, I_α^8 and I_α^9

Table of the generated parametric implications

Table: Table of Parametric Implications Via Classes of Strong Negations

Parametric Implications
$I_{\lambda}^{1}(x,y)=\frac{1-\min\left\{x,\frac{1-y}{1+\lambda\cdot y}\right\}}{1+\lambda\cdot\min\left\{x,\frac{1-y}{1+\lambda\cdot y}\right\}}$
$I_{\lambda}^{2}(x,y)=\frac{1-x\cdot\frac{1-y}{1+\lambda\cdot y}}{1+\lambda\cdot x\cdot\frac{1-y}{1+\lambda\cdot y}}$
$I_{\lambda}^{3}(x,y)=\frac{1-\max\left\{x+\frac{1-y}{1+\lambda\cdot y}-1,0\right\}}{1+\lambda\cdot\max\left\{x+\frac{1-y}{1+\lambda\cdot y}-1,0\right\}}$
$I_{\omega}^{4}(x,y)=\sqrt[\omega]{1-\left(\min\left\{x,\ \sqrt[\omega]{1-y^{\omega}}\right\}\right)^{\omega}}$
$I_{\omega}^{5}(x,y)=\sqrt[\omega]{1-x^{\omega}+(x\cdot y)^{\omega}}$
$I_{\omega}^{6}(x,y)=\sqrt[\omega]{1-\left(\max\left\{x+\ \sqrt[\omega]{1-y^{\omega}}-1,0\right\}\right)^{\omega}}$
$I_{a}^{7}(x,y)=\sqrt{(a^2-1).\left(\min\left\{x,\sqrt{(a^2-1)y^2+1}+\alpha\cdot y\right\}\right)^2+1}+a\cdot\min\left\{x,\sqrt{(a^2-1)y^2+1}+\alpha\cdot y\right\}$
$I_{a}^{8}(x,y)=\sqrt{(a^2-1)\cdot x^2\cdot\left(\sqrt{(a^2-1)\,y^2+1}+\alpha\cdot y\right)^2+1}+a\cdot x\cdot\left(\sqrt{(a^2-1)\,y^2+1}+\alpha\cdot y\right)$
$I_{a}^{9}(x,y)=\sqrt{(a^2-1).\left(\max\left\{x+\sqrt{(a^2-1)y^2+1}+\alpha\cdot y-1,0\right\}\right)^2+1}+a\cdot\max\left\{x+\sqrt{(a^2-1)y^2+1}+\alpha\cdot y-1,0\right\}$

Conclusion

The field of fuzzy implications has been surveyed by many researchers, resulting in the creation of many strategies for the generalization of fuzzy implications.

Conclusion

Utilizing the classes of strong fuzzy negations (Suggeno, Yager and Souliotis-Papadopoulos) and the strict fuzzy t-norms (T-minimum, T-product and T-Lukasiewicz), a strong algorithmic procedure of finding a parametric fuzzy implications was produced. This comprises the result of the research. In other words, a mathematical tool for the achievement of approximate reasoning was created.

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Thank you!!!