Application of Algorithmic Fuzzy Implications on Climatic Data

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Abstract. In this paper we present a new Fuzzy Implication Generator via Fuzzy Negations which was generated via conical sections, in combination with the well-known Fuzzy Conjunction T-norm = min. Among these implications we choose the most appropriate one, after comparing them with the empiristic implication, which was created with the help of real temperature and humidity data from the Hellenic Meteorological Service. The use of the empiristic implication is based on real data and also it reduces the volume of the data but without cancelling them. Finally, the pseudo-code, which was used in the programming part of the paper, uses the new Fuzzy Implication Generator and approaches the empiristic implication satisfactorily which is our final goal.

Keywords: Fuzzy implication · Empiristic implication · Fuzzy negation via conical sections

1 Introduction

The Theory of Fuzzy Implications and Fuzzy Negations plays an important role in many applications of fuzzy logic, such as approximate reasoning, formal methods of proof, inference systems, decision support systems (cf [2] and [5]). Recognizing the above important role of Fuzzy Implications and Fuzzy Negations, we tried to construct Fuzzy implications from Fuzzy negations, so that we could change the implication with one parameter, thus giving an algorithmic procedure of the "if......then......" rule. The tools for this construction, namely a fuzzy implication generator, were mainly the conclusions of (cf [11]) and in particular the following formula 1

$$N(x) = \sqrt{(a^2-1)x^2+1} + ax, \quad x \in [0,1], \quad a \leq 0. \tag{1}$$

producing fuzzy negations via conical sections and Corollary 2.5.31. (see [1]). The combination of these two through the fuzzy conjunction T-norm = min gave us an algorithmic procedure for the evaluation of the best implication with respect to the problem data. For the evaluation of the best implication we used

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the empiristic implication (see [8]) as a comparison measurement. The empiristic implication does not satisfy any of the properties of the fuzzy implications and has no specific formula, meaning it is not a function. However, the choice of this particular implication was not random as it was based on its ability to be directly calculated by the empirical data without being affected by the amount of the data.

The paper follows the following structure: The section Preliminaries presents the theoretical background of the paper, such as the definitions of the Fuzzy Implications, Fuzzy Negations and Triangular norms. The section Main Results shows the implication which is constructed by a strong fuzzy negation and a t-norm with $T_M(x, y) = \min\{x, y\}$. Next, the empiristic implication and the algorithm for its calculation are presented. The data used are real, such as the average monthly temperature and the average monthly relative humidity, two of the most important climatic variables in meteorology (see [4]). Furthermore, the algorithmic process for finding the best fuzzy implication among the empiristic implication, the three known implications from the literature (Kleen-Dienes, Lukasiewicz, Reichenbach) (see [1]) and the constructed parametric implication is analysed. The calculation of the implications with the use of data was done in the Matlab programming environment. We present the documentation of the Matlab code which was used to calculate the implications.

2 Preliminaries

The following short theoretical background is important in order to understand this paper.

2.1 Fuzzy Implication

In the literature we can find several different definitions of fuzzy implications. In this paper we will use the following one, which is equivalent to the definition proposed by Kitainik [6], (see also [3] and [1]).

Definition 1. *A function $I : [0, 1]x[0, 1] \to [0, 1]$ is called a fuzzy implication if for all $x, x_1, x_2, y, y_1, y_2 \in [0, 1]$ the following conditions are satisfied:*
$(I1)$ $x_1 \leq x_2$ *then* $I(x_1, y) \geq I(x_2, y)$, *i.e,* $I(\cdot, y)$ *is decreasing,*
$(I2)$ $y_1 \leq y_2$ *then* $I(x, y_1) \leq I(x, y_2)$, *i.e.,* $I(x, \cdot)$ *is increasing,*
$(I3)$ $I(0, 0) = 1$
$(I4)$ $I(1, 1) = 1$
$(I5)$ $I(1, 0) = 0$

Example 1. *Some examples of Fuzzy Implications are given below:*
Kleene-Dienes: $I_{KD}(x, y) = \max\{1 - x, y\}$
Lukasiewicz: $I_{LK}(x, y) = \min\{1, 1 - x + y\}$
Reichenbach: $I_{RC}(x, y) = 1 - x + x \cdot y$

2.2 Fuzzy Negations

The following definitions and examples can be found [1,3,9], and [10].

Definition 2. *A function* $N : (0, 1) \to [0, 1]$ *is called a Fuzzy negation if*
$(N1)$ $N(0) = 1,\ N(1) = 0$
$(N2)$ N *is decreasing*

Definition 3. *A fuzzy negation N is called strict if, in addition,*
(N3) N is strictly decreasing,
(N4) N is continuous,
A fuzzy negation N is called strong if the following property is met,
$(N5)$ $N(N(x)) = x,\ x \in [0, 1]$

Example 2. *Examples of Fuzzy Negations are given below.*
$N_K(x) = 1 - x^2$, *strict*
$N_R(x) = 1 - \sqrt{x}$, *strict*
$N^{\lambda}(x) = \frac{1-x}{1+\lambda x},\ \lambda \in (-1, +\infty)$ *strong Sugeno class*
$N^{W}(x) = (1 - x^{w})^{\frac{1}{w}},\ w \in (0, +\infty)$ *strong Yager class*

Remark 1. The paper [11] proves a new family of strong fuzzy negations, which is produced by conical sections and is given from the Eq. (1), which will play a key role in building the algorithmic procedure we propose in the section Main Results.

2.3 Triangular Norms (Conjunctions)

The Triangular norms were introduced by Menger [9] and were later reconstructed by Schweizer and Sklar [10] in the form they have today. In essence, they are a generalization of the classical binary conjunction (∧) into a fuzzy intersection. The following definition can be found in the monograph by Klement et. al [7], (see also [1]).

Definition 4. *A function* $T : [0, 1]^2 \to [0, 1]$ *is called triangular norms shortly t- norm, if it satisfies, for all* $x, y \in [0, 1]$*, the following conditions:*
$(T1)$ $T(x, y) = T(y, x)$ $(commutativity)$
$(T2)$ $T(x, T(y, z)) = T(T(x, y), z)$
$(T3)$ $if\ \ y \leq z,\ then\ T(x, y) \leq T(x, z)$ $(monotonicity)$
$(T4)$ $T(x, 1) = x$ $(boundary\ condition)$

Table 1 lists a few of the common t-norms.

In the paper we will use the most basic of all t-norms, which is the minimum

$$T_M(x, y) = \min\{x, y\} \tag{2}$$

Table 1. Table basic t-norms

Minimum	$T_M(x,y) = \min\{x,y\}$
Algebraic product	$T_p(x,y) = x \cdot y$
Lukasiewicz	$T_{LK}(x,y) = \max(x+y-1,0)$
Active product	$T_D(x,y) = \begin{cases} 0 & x,y \in [0,1) \\ min(x,y) & otherwise \end{cases}$
Nilpotent minimum	$T_{nM}(x,y) = \begin{cases} 0 & x+y \leq 1 \\ min(x,y) & otherwise \end{cases}$

3 Main Results

3.1 Construction of Fuzzy Implications via Strong Negations

The main purpose of this work is to create a two plays function that satisfies Definition 1, utilizing the temperature and humidity data given by the Hellenic National Meteorological Service. That is, the construction of an implication that gives the degree of truth of the two variables, the temperature and the humidity. To achieve this, the two variables are normalized with the help of fuzzy sets. In this way the temperature gets values of [0, 1] and the humidity gets values of [0, 1], which is the degree of truth of the two variables. For example, if the temperature between $[21°, 31°]$ degrees is considered high, then it has a degree of truth 1. And similarly if the humidity between [40%, 50%] is considered low, then it has a degree of truth of 0.7. Our goal is to construct an implication that gives the degree of truth, as does the statement below:

"If the temperature is high, then the humidity is low."

To what degree of truth can we respond to this statement?

This is our intention, that is, to find an implication that is close enough to the correlation of the two variables, the temperature and the humidity, as are shown in our data. But to find such an implication there must be a comparison measure, that is, we want from our data to ensure the degree of truth of the temperature and the humidity pair as they are given. The comparison measure in the present work is the empiristic implication that uses all the data in order to produce a table in which in the first row and the first column there will be the data grouped into classes and in each cell there will be the corresponding degree of truth of the data. Then, we compare each of the Kleen-Dienes, Lukasiewicz, Reichenbach (see [1]) and the parametric implications which will be generated, with the empiristic one, using the square error of the difference of the aforementioned implication tables from the empiristic implication table.

The smallest square error will give the best implication.

In book [1], and in particular in Corollary (2.5.31), the implication generated by a strong fuzzy negation and a t-norm is examined and the formula

$$I(x,y) = N(T(x,N(y)), \quad x,y \in [0,1] \tag{3}$$

is proposed. Using the above implication (3) (see 3) with t-norm (2) (see 2) and the formula (1) (see 1) with parameter α and after the appropriate calculations, the following equation occurs.

$$I(x,y) = \sqrt{(a^2-1)\,.\left(\min\left(x, \sqrt{(a^2-1)\,y^2+1}+ay\right)\right)^2+1} + a.\min\left(x, \sqrt{(a^2-1)\,y^2+1}+ay\right), y \in [0,1], a \leq 0 \qquad \text{(equation a)}$$

The above implication of (equation a), which is a new generator fuzzy implications, is important because it has the parameter α which helps us to use the implication on our data and at the same time examine for which value of α we have the best approach. Hence, an algorithmic process of finding a better implication is created which will play an important role in the course of the paper.

3.2 Empiristic Implication

In order to be able to estimate which of the proposed implications approaches the pairs (Temperature, humidity) of the Hellenic Meteorological Service, we need to have a comparison measurement. In this paper we use the empiristic implication as a measure see [8]. The empiristic implication will be presented while explaining the steps of the algorithm, based on our data, which derive from the Hellenic Meteorological Service and are the average monthly temperature and the average monthly humidity of the last five years from the 13 regions of Greece (see Fig. 1, Fig. 2).

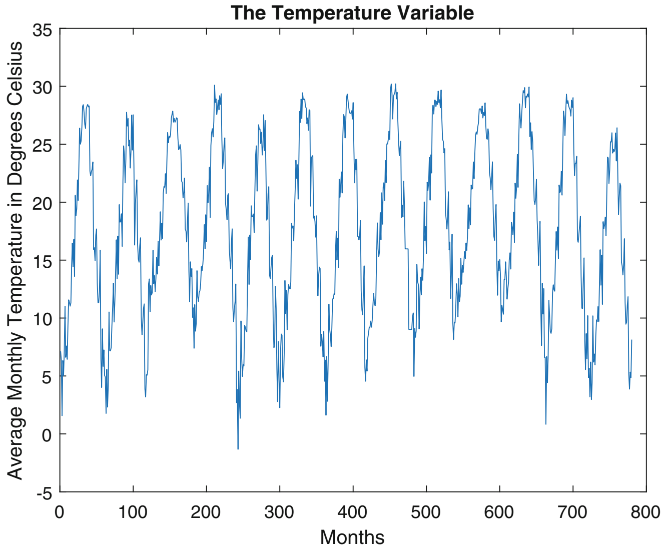

Fig. 1. Temperature variable

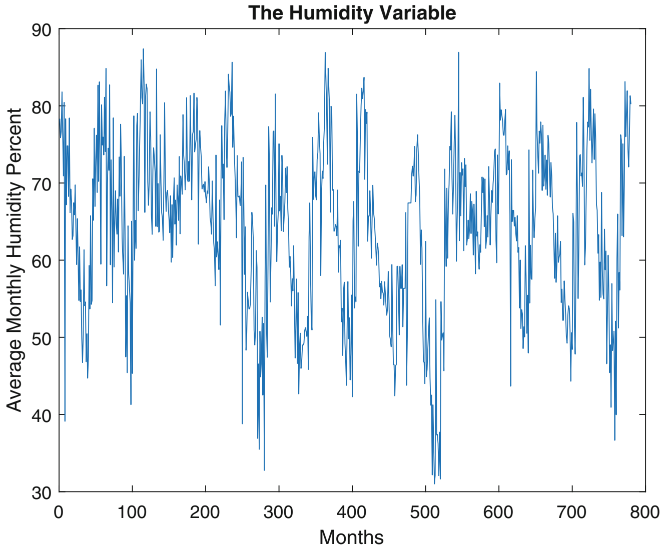

Fig. 2. The humidity variable

Our goal is to find which implication approaches best our data dependence (temperature and humidity), giving us as a result the degree of the true coexistence of the two corresponding values. For example, we would like to see the degree of truth of the statement:

if the temperature is 20 °C then the humidity is 60%

First, we find the table representing the empiristic implication. For the construction of the empiristic implication, we divide the 780 temperature and relative humidity data into 11 classes with the use of the Sturges type

$$c = 1 + \log_2 n \overset{n=780}{\Leftrightarrow} c = 1 + \log_2(780) \Leftrightarrow c = 1 + \frac{\log(780)}{\log(2)} \Leftrightarrow c = 10.6 \tag{4}$$

after first placing them in ascending order. Each class has its median as a representative. This is how we create the empiristic implication table, which has the medians of the humidity classes in the first row while the medians of the temperature classes are in the first column. Each cell of the table is divided by the sum of the column in which it belongs. In this way, we have in each cell the degree of the truth of the coexistence of the values of the corresponding column and row of the cell (see Table 2). Then we normalize the temperature and the humidity medians. Next, we check first whether the three known implications (Kleen-Dienes, Lukasiewicz, Reichenbach) (see [1]) approach the table above. Then, we examine the norm of each of the above implications with the empirical implication. The results are as follows.

Table 2. Table of the empiristic implication

0.0282	0	0.0141	0.0423	0.0282	0.0563	0.1127	0.1127	0.1831	0.1972	0.2286
0.0141	0	0.0141	0.0704	0.0563	0.0845	0.1972	0.0704	0.1268	0.2113	0.1571
0	0	0	0.0563	0.0423	0.1127	0.0986	0.1408	0.1127	0.1408	0.3000
0	0.0141	0.0423	0.0563	0.0704	0.1408	0.0986	0.1408	0.1831	0.1690	0.0857
0.0141	0.0423	0.0845	0.0423	0.1268	0.0986	0.0845	0.1549	0.1268	0.1127	0.1143
0.0423	0.0282	0.1408	0.0845	0.1972	0.0563	0.0423	0.1690	0.0563	0.1127	0.0714
0.0563	0.0563	0.1268	0.1268	0.1268	0.1690	0.1127	0.0845	0.0704	0.0423	0.0286
0.0845	0.1268	0.1831	0.1972	0.0845	0.0986	0.0845	0.0423	0.0704	0.0141	0.0143
0.1127	0.2254	0.1408	0.1127	0.1127	0.1268	0.0845	0.0282	0.0563	0	0
0.2676	0.1690	0.1549	0.1549	0.0704	0.0423	0.0704	0.0563	0.0141	0	0
0.3803	0.3380	0.0986	0.0563	0.0845	0.0141	0.0141	0	0	0	0

The Results of the Norms. The squared error of the two implications (empiristic and Kleen-Dienes) gives the result is 6.2465.

The squared error of the two implications (empiristic and Kleen-Dienes) gives the result is 8.7386.

The squared error of the two implications (empiristic and Kleen-Dienes) gives the result is 7.4448.

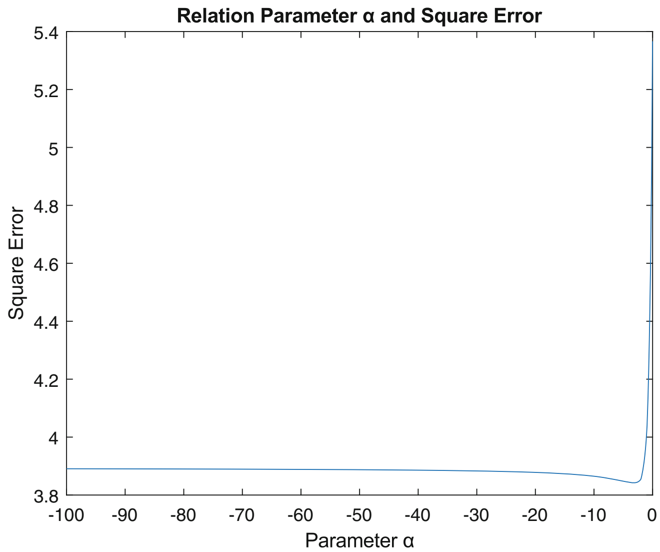

Fig. 3. Relation parameter α and square error.

After the examination of the three implications is completed, we proceed to find the best parameter $a \leq 0$ that we have from the (equation a). We have the best approach for the value $\alpha = -3$ and with a very good approach the squared error of the two implications (empiristic and Kleen-Dienes) gives the result is 3.8425 (see Fig. 3).

Remark 2. To achieve our goal, that is, to approach the empiristic implication table, we used 3 linguistic variables (low, medium and high) for temperature and humidity respectively.
$[a, b, c, d] = [-1.33\ -1.33\ 7\ 12]$ is low temperature
$[a, b, c, d] = [10\ 13\ 15\ 18]$ is medium temperature
$[a, b, c, d] = [16\ 21\ 30.21\ 30.21]$ is high temperature
$[a, b, c, d] = [31.01\ 31.01\ 40\ 45]$ is low humidity
$[a, b, c, d] = [43\ 50\ 60\ 65]$ is medium humidity
$[a, b, c, d] = [64\ 75\ 87.39\ 87.39]$ is high humidity
(see Fig. 4, Fig. 5). Also, to avoid the property of the implication I (0,1) = 1, which reinforces the falsehood, we tried to obtain the values of $x \neq 0$.

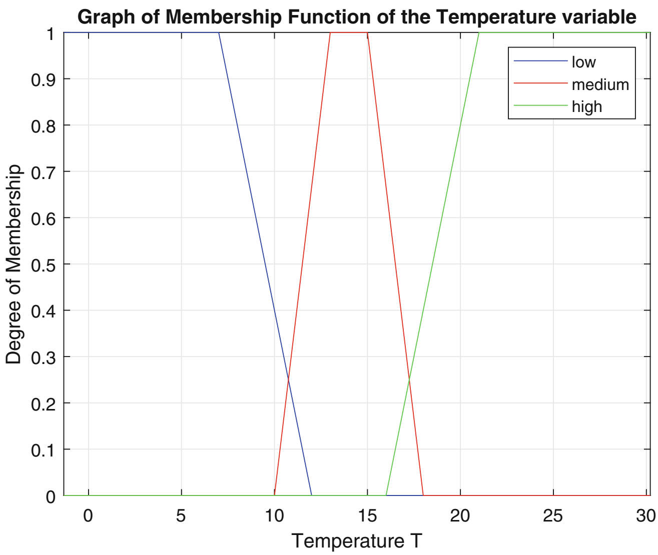

Fig. 4. Membership function of the temperature.

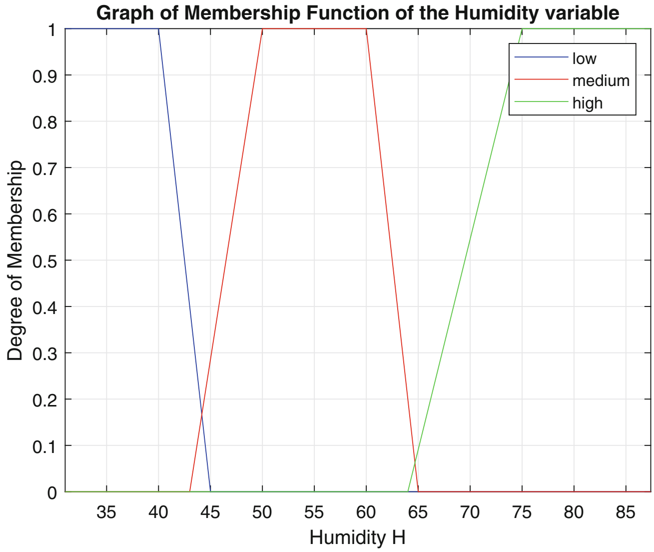

Fig. 5. Membership function of the humidity.

3.3 The Documentation of the Matlab Code

The data are in the file Data.xlsx. Our application includes a case study, which uses real climatic data (average monthly temperature and average monthly relative humidity) of the last five years 2015–2019 from the 13 regions of Greece for the 12 months of each year and evaluates the empiristic implication, with fuzzy implications we created. The application was implemented in Matlab R2018b and includes the steps:

1. We load the data onto the program, which creates two 780×1 tables. The lines in the tables are the 780 observations and the columns in the tables are the temperature and relative humidity variables. Temperature, Humidity.
2. We find the minimum and maximum values of the columns that make up the range of the variables.
3. We have the original table with the first column is X and the second is Y.
4. We add to this table a column which is the increment number in order not to miss the original pairs (xi, yi). The column we add is the third one.
5. Later, we create a different table for X together with its increment number and a different one for Y. Then, we have the initial position for each X and Y so we sort in ascending order according to the values. We notice that in the first column we have X and Y in ascending order and in the second column their position in the original data.
6. We normalize using trapezoidal membership functions
7. We apply the Sturges rule: (see 4) in order to divide the sorted data columns in classes.

8. Next, we create 11 classes for each sorted table.
9. We add the classes to the third column in the sorted tables. The format of the sorted tables is: the first column has X in ascending order, the second column has the initial position and the third column has the class.
10. Then, we sort the above two tables by increment number to get the data back to their original position along with their classes.
11. We put the above tables in a table where the first column is X, the second is the class of X, the third is Y and the fourth is the class of Y.
12. We create a Zero Table. The table will be a column larger and a column smaller to place the medians.
13. Finally, we create the table we want without adding values to the first row and column.
14. We create medians for the classes of X and place them in the Final Table.
15. We create medians for the classes of Y and place them in the Final Table.
16. The Final Table has the medians of the classes of Y as its first row and the medians of the classes of X as its first column.
17. We create a new table who has the first row and the first column with medians.
18. We form the rest of the table by calling the function of parametric implication.
19. The table imp1 is the table of the empiristic implication.
20. We will check three well-known implications with the data we have.
21. The first is Kleen-Dienes (see Examples1.)
22. Table A is an 11×1 column table containing the temperature medians.
23. Table B is a 1×11 row table containing the humidity medians.
24. Table A1 is an 11×1 column table containing the nondimensionalized values of the temperature medians.
25. Table B1 is a 1×11 row table containing the nondimensionalized values of humidity medians.
26. The final table of Kleen-Dienes implication is imp2 and the control of the norm of the two implications (empiristic and Kleen-Dienes) is nor1 = norm (imp1-imp2).
27. The second implication is Lukasiewicz (see Examples1.)
28. The final table of the Lukasiewicz implication is imp3 and the control of the norm of the two implications (empiristic and Lukasiewicz) is nor2 = norm (imp1-imp3).
29. The third implication is Reichenbach (see Examples1.)
30. The final table of Reichenbach's implication is imp4 and the control of the norm of the two implications (empiristic and Reichenbach) is nor3 = norm (imp1-imp4).
31. The final table of Parametric's implication is OtherTable and the control of the norm of the two implications (empiristic and Parametric) is nor4 = norm (imp1-OtherTable).

4 Conclusions

It is now evident that the type of the strong fuzzy negations that are generated via conical sections, combined with the fuzzy conjunction (T-norm = min), give us a robust algorithmic process of finding the more appropriate fuzzy implication. So we see that a purely mathematical process and even a geometric one is a powerful tool when it is well supported to achieve approximate reasoning. In addition, a thorough and careful study of the data in the correct order will greatly reduce the computational complexity. Our future research on the applications of fuzzy implications will continue with the aim of achieving better results of the convergence of the empiristic implication and the implications the strong fuzzy negations that are generated via conical sections.

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References

1. Baczynski, M., Jayaram, B.: Fuzzy Implications. Springer-Verlag, Heidelberg (2008). https://doi.org/10.1007/978-3-540-69082-5_1
2. Fodor, J.C.: Contrapositive symmetry of fuzzy implications. Fuzzy Sets Syst. **69**, 141–156 (1995)
3. Fodor, J.C., Roubens, M.: Fuzzy preference Modelling and Multicriteria Decision Support. Kluwer, Dordrecht (1994)
4. Hellenic National Meteorological Service. http://www.hnms.gr/emy/el/climatology/climatology_month
5. Jenei, S.: A new approach for interpolation and extrapolation of compact fuzzy quantities. The one dimensional case. In: Klement, E.P., Stout, L.N. (eds.) Proceedings of the 21th Linz Seminar on Fuzzy Set Theory, Linz, Austria, pp. 13–18 (2000)
6. Kitainik, L.: Fuzzy Decision Procedures with Binary Relations. Kluwer, Dordrecht (1993)
7. Klement, E.P., Mesiar, R., Pap, E.: Triangular Norms. Kluwer, Dordrecht (2000)
8. Mattas, K., Papadopoulos, B.: Fuzzy empiristic implication, a new approach. In: Mattas, K., Papadopoulos, B. (eds.) Modern Discrete Mathematics and Analysis, SOIA. Springer Optimization and Its Applications, vol. 131, pp. 317–331 (2018)
9. Menger, K.: Statistical metrics. Proc. Nat. Acad. Sci. USA **28**, 535–537 (1942)
10. Schweizer, B., Sklar, A.: Probabilistic Metric Spaces. North-Holland, New York (1983)
11. Souliotis, G., Papadopoulos, B.: An algorithm for producing fuzzy negations via conical sections. Algorithms **12**(5), 89 (2019). https://doi.org/10.3390/a12050089